https://doi.org/10.1016/j.exmath.2015.12.002

THE CENTER PROBLEM AND COMPOSITION CONDITION FOR ABEL DIFFERENTIAL EQUATIONS

JAUME GINÉ, MAITE GRAU AND XAVIER SANTALLUSIA

ABSTRACT. The classical Poincaré center-focus problem for planar polynomial systems of ordinary differential equations can be transformed, in certain particular cases, to the center problem of a trigonometric Abel differential equation. Several research papers focused on the study of the center problem for trigonometric Abel differential equations. Polynomial Abel differential equations are also considered in the literature as a model problem. In this work we make a survey of the most important results in this context and we provide the state of the art of several related conjectures. We give two new results on these conjectures.

1. Introduction

Consider a planar differential system

$$\dot{x} = -y + P(x,y), \qquad \dot{y} = x + Q(x,y), \tag{1}$$

where the dot denotes derivation with respect to an independent real variable t, x and y are real and where P and Q are real analytic functions without constant nor linear terms. We recall that a singular point is a center if in a neighborhood of the singular point all the solutions are periodic. In this paper we only consider the singular point at the origin of coordinates in system (1). The *center problem* consists in determining necessary and sufficient conditions on P and Q such that system (1) has a center at the origin.

In the particular case that P and Q are homogeneous polynomials system (1) can be transformed into an Abel trigonometric differential equation. In case P and Q are homogeneous polynomials of degree n, with $n \geq 2$, the process is to take polar coordinates (r, θ) and system

2010 *Mathematics Subject Classification.* Primary 34C25. Secondary 34C07.

Key words and phrases. Abel equations, center problem, composition condition, moment conditions, composition conjecture.

The authors are partially supported by a MINECO/FEDER grant number MTM2014-53703-P and by an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204.

(1) becomes

$$\dot{r} = f(\theta)r^n, \qquad \dot{\theta} = 1 + g(\theta)r^{n-1},$$

where

$$f(\theta) = P(\cos\theta, \sin\theta)\cos\theta + Q(\cos\theta, \sin\theta)\sin\theta,$$
$$g(\theta) = Q(\cos\theta, \sin\theta)\cos\theta - P(\cos\theta, \sin\theta)\sin\theta.$$

Now applying the Cherkas transformation [14] given by

$$\rho = \frac{r^{n-1}}{1 + g(\theta)r^{n-1}} \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(n-1)}}{(1 - \rho g(\theta))^{1/(n-1)}},$$

system (1) becomes the Abel trigonometric differential equation

$$\frac{d\rho}{d\theta} = ((1-n)f(\theta) + g'(\theta))\,\rho^2 + ((1-n)f(\theta)g(\theta))\,\rho^3.$$

By the regularity of the Cherkas transformation and its inverse at $r = \rho = 0$, system (1) has a center at the origin if and only if the former ordinary differential equation has a center. Hence we have transformed the center-focus problem of system (1) into a center problem for an Abel differential equation. Other examples of systems of the form (1) which can be transformed into an Abel differential equation can be found in [18].

In this context a *trigonometric Abel differential equation* is an ordinary differential equation of the form

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{2}$$

where ρ is real, θ is a real and periodic independent variable with $\theta \in [0, 2\pi]$, and $a_1(\theta)$ and $a_2(\theta)$ are real trigonometric polynomials. We recall that the *center problem for a trigonometric Abel differential equation* (2) is to characterize when all the solutions in a neighborhood of the solution $\rho = 0$ are periodic of period 2π.

Some authors also consider polynomial Abel differential equations as a model to tackle the center problem for a trigonometric Abel differential equation, see [7, 8, 9]. We denote as a *polynomial Abel differential equation* an ordinary differential equation of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \tag{3}$$

where y is real, x is a real independent variable considered in a real interval $[a, b]$ and $p(x)$ and $q(x)$ are real polynomials in $\mathbb{R}[x]$. The *center problem for a polynomial Abel equation* (3) is to characterize when all the solutions in a neighborhood of the solution $y = 0$ take the same value when $x = a$ and $x = b$, i.e. $y(a) = y(b)$. In this framework, given any real continuous function $c(x)$, we denote by $\tilde{c}(x) := \int_a^x c(\sigma)d\sigma$ and we will say that a real continuous function $w(x)$ is *periodic in* $[a, b]$ if $w(a) = w(b)$.

Alwash and Lloyd in [4] provided a sufficient condition for an equation (2) to have a center in $[0, 2\pi]$. Inspired by this work, Briskin, Françoise and Yomdin in [7] provided the following sufficient condition for the polynomial Abel equation (3).

Theorem 1. [7] *If there exists a real differentiable function w periodic in $[a, b]$ and such that*

$$\tilde{p}(x) = p_1(w(x)) \quad \text{and} \quad \tilde{q}(x) = q_1(w(x))$$

for some real differentiable functions p_1 and q_1, then the polynomial Abel equation (3) *has a center in* $[a, b]$.

In [17] it is shown that if the sufficient condition stated in Theorem 1 is satisfied then there is a countable set of definite integrals which need to vanish. In [17] it is also shown that this is equivalent to the existence of a real polynomial $w(x)$ with $w(a) = w(b)$ and two real polynomials $p_1(x)$ and $q_1(x)$ such that $\tilde{p}(x) = p_1(w(x))$ and $\tilde{q}(x) = q_1(w(x))$. This sufficient condition is known as the *composition condition*.

To see that the composition condition implies that equation (3) has a center in $[a, b]$ one can consider the transformation $y(x) = Y(w(x))$ in equation (3) in order to obtain the following Abel differential equation

$$\frac{dY}{dw} = p_1'(w)Y^2 + q_1'(w)Y^3. \tag{4}$$

Hence, there is a bijection between the solutions $Y = Y(w)$ of equation (4) and the solutions $y = Y(w(x))$ of equation (3). Since w is periodic in $[a, b]$, we get that equation (3) has a center in $[a, b]$ because $y(a) = Y(w(a)) = Y(w(b)) = y(b)$.

It turns out that all the known polynomial Abel differential equations which have a center in $[a, b]$ satisfy the composition condition. The *composition conjecture*, see Conjecture 3, is that the sufficient condition given in Theorem 1 is also necessary. That is, if a polynomial Abel equation (3) has a center in $[a, b]$, the conjecture states that the composition condition is satisfied.

For a trigonometric Abel differential equation (2), Alwash in [1] showed that this conjecture is not true, see also [3, 15]. The *composition condition* for a trigonometric Abel differential equation (2) is that there exist real polynomials $p_1(x), p_2(x) \in \mathbb{R}[x]$ and a trigonometric polynomial $\omega(\theta)$ such that $\tilde{a}_i(\theta) = p_i(\omega(\theta))$, for $i = 1, 2$. Recall that $\tilde{a_i}(\theta) := \int_0^\theta a_i(s)ds$. The fact that $\omega(\theta)$ and p_1, p_2 can be taken to be polynomials is proved in [17, 20]. There exist several counterexamples of the fact that the composition conjecture is not satisfied in the trigonometric case. The authors of [1, 3, 15] provide examples of trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ for which the corresponding trigonometric Abel differential equation (2) has a center and does not verify the composition condition.

The paper is organized as follows. The following section contains a summary of some conjectures related to the composition conjecture and the corresponding results. Section 3 is devoted to the known results about the composition conjecture together with two new statements, cf. Theorems 5 and 7. These statements are proved in sections 4 and 6, respectively. The last section 7 contains an appendix with the code of two programs, written in the language of Mathematica and used in these proofs.

2. Some other composition conjectures

In this section we consider the polynomial Abel differential equation

$$\frac{dy}{dx} = p(x)y^2 + \varepsilon q(x)y^3, \tag{5}$$

where y is real, x is a real variable considered on the real interval $[a, b]$, $\varepsilon \in \mathbb{R}$ and $p(x)$ and $q(x)$ are real polynomials. We also assume that $\int_a^b p(s)ds = 0$.

One of the problems that can be tackled is to characterize when equation (5) has a center in $[a, b]$ for all ε with $|\varepsilon|$ small enough. This type of centers are called *infinitesimal centers* or *persistent centers*, see [3, 15].

The following computations were first performed in [7]. We include them for the sake of completeness. Given real values ε and y_0, we denote by $Y_\varepsilon(x; y_0)$ the solution of equation (5) for the value of the parameter ε and with initial condition y_0, that is, the real function $Y_\varepsilon(x; y_0)$ satisfies

$$\frac{\partial}{\partial x} Y_\varepsilon(x; y_0) = p(x)Y_\varepsilon(x; y_0)^2 + \varepsilon q(x)Y_\varepsilon(x; y_0)^3, \quad Y_\varepsilon(a; y_0) = y_0. \tag{6}$$

We remark that, with this notation, a persistent center is when $Y_\varepsilon(b; y_0) = y_0$ for all ε with $|\varepsilon|$ small enough and for all y_0 with $|y_0|$ small enough.

Recall that we denote $\tilde{p}(x) = \int_a^x p(s)ds$. We note that when $\varepsilon = 0$, equation (5) has a center in $[a, b]$ because

$$Y_0(x; y_0) = \frac{y_0}{1 - y_0\tilde{p}(x)},$$

and clearly for all $|y_0| < \mu_p$, where

$$\mu_p := \min_{x \in [a,b]} \frac{1}{|\tilde{p}(x)|},$$

we have that $Y_0(x; y_0)$ is continuous in $[a, b]$ and $Y_0(a; y_0) = Y_0(b; y_0) = y_0$ due to the assumption $\tilde{p}(b) = \int_a^b p(s)ds = 0$. Note that $\mu_p > 0$ and therefore all the solutions in a neighborhood of $y = 0$ are defined for all $x \in [a, b]$. Equation (5) with $\varepsilon = 0$ has a center in $[a, b]$. As we have said, this means that there exists a family of periodic orbits in $[a, b]$ for equation (5) with $\varepsilon = 0$ in a neighborhood of the solution $y = 0$. The underlying idea when considering equation (5) is to determine which orbits in this family persist for values of ε with $|\varepsilon|$ small enough. Since the dependence of equation (5) in ε is analytic (indeed linear), we have that the dependence of $Y_\varepsilon(x; y_0)$ in ε is analytic. Thus, we can develop this function in ε in a neighborhood of $\varepsilon = 0$ as

$$Y_\varepsilon(x; y_0) = Y_0(x; y_0) + \pi_1(x; y_0)\varepsilon + o(\varepsilon).$$

Since, from (6), $Y_\varepsilon(a; y_0) = y_0$ and $Y_0(a; y_0) = y_0$, we deduce that $\pi_1(a; y_0) = 0$. Indeed, we can develop the first equation of (6) in powers of ε and equating the coefficients of ε^1 we deduce that

$$\frac{\partial}{\partial x}\pi_1(x; y_0) = 2p(x)Y_0(x; y_0)\pi_1(x; y_0) + q(x)Y_0(x; y_0)^3.$$

Integrating this linear ordinary differential equation for $\pi_1(x; y_0)$ we get that

$$\pi_1(x; y_0) = \frac{y_0^3}{(1 - y_0\tilde{p}(x))^2} \int_a^x \frac{q(\sigma)}{1 - y_0\tilde{p}(\sigma)}\, d\sigma. \tag{7}$$

Therefore, the necessary and sufficient condition for equation (5) to have a center in $[a, b]$ *at first order in* ε is that $\pi_1(b; y_0) = 0$. From (7), we deduce that this is to say that

$$\int_a^b \frac{q(\sigma)}{1 - y_0\tilde{p}(\sigma)}\, d\sigma \equiv 0,$$

for all y_0 with $|y_0|$ close enough to 0. We can develop this integral in powers of y_0 in a neighborhood of $y_0 = 0$ and we get that this condition is equivalent to

$$\int_a^b q(\sigma)\, \tilde{p}^{\,n}(\sigma)\, d\sigma = 0, \tag{8}$$

for all natural numbers $n \in \mathbb{N} \cup \{0\}$, see [3]. Conditions (8) are called the *moment conditions*. The *composition conjecture for moments* is that the moments conditions imply the composition condition. Moreover, in [10] it is proved that "at infinity" the center conditions are reduced to the moment conditions.

A counterexample to the composition conjecture for moments in the polynomial case was given in [25]. We reproduce here this example, see also [2, 3, 15].

In equation (5) we take $p(x) = T_6'(x)$ and $q(x) = T_2'(x) + T_3'(x)$ where $T_i(x)$ denotes the i-th Chebyshev polynomial and $T_i'(x)$ its derivative. We have that $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$ and $T_6(x) = (T_3 \circ T_2)(x) = (T_2 \circ T_3)(x) = 32x^6 - 48x^4 + 18x^2 - 1$. We take also $a = -\sqrt{3}/2$ and $b = \sqrt{3}/2$. Under these conditions the moment conditions (8) are zero taking into account that $T_2(\sqrt{3}/2) - T_2(-\sqrt{3}/2) = T_3(\sqrt{3}/2) - T_3(-\sqrt{3}/2) = 0$. We note that if an equation (5) satisfies the composition condition then the moment conditions are satisfied. Indeed, if an equation (5) satisfies the composition condition then the following conditions

$$\int_a^b p(\sigma)\tilde{q}^{\,n}(\sigma)d\sigma = 0, \tag{9}$$

are satisfied for all natural numbers $n \in \mathbb{N} \cup \{0\}$. This is due to the fact that if $p(x)$ and $q(x)$ satisfy the composition condition then the integrands of the integrals (8) and (9) are functions of $w(\sigma)$ multiplied by $w'(\sigma)$ and since $w(\sigma)$ is periodic in $[a, b]$, we deduce that they all need to be zero. Now we see that there are integrals in (9) for this example that are not zero. For instance,

$$\int_a^b p(\sigma)\tilde{q}^{\,2}(\sigma)d\sigma = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_6'(\sigma)(T_2(\sigma) + T_3(\sigma))^2 d\sigma \neq 0.$$

Hence, equation (5) with $p(x) = T_6'(x)$ and $q(x) = T_2'(x) + T_3'(x)$ does not satisfy the composition condition.

We even have a stronger result. If one considers the differential equation of this example with $\varepsilon = 1$, this equation does not have a center in $[-\sqrt{3}/2, \sqrt{3}/2]$. Easy computations show that the sixth Poincaré–Liapunov constant is $v_6 = 432\sqrt{3}/385$. We have used the method explained in section 4 to compute the Poincaré–Liapunov constants v_2, v_3, v_4, v_5 and v_6. Therefore, this example shows that if the equation has a center at first order of ε, then it is not necessary that the equation has a center when $\varepsilon = 1$.

In the trigonometric case, that is, if one considers a trigonometric Abel differential equation of the form

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + \varepsilon a_2(\theta)\rho^3, \tag{10}$$

where ρ is real, θ is a real and periodic independent variable with $\theta \in [0, 2\pi]$ and ε is a real value close to 0, one can define the composition conjecture for moments analogously to the polynomial case. The moment conditions in this case write as

$$\int_0^{2\pi} \tilde{a}_1^n(\theta) a_2(\theta) d\theta = 0, \tag{11}$$

with $n \in \mathbb{N} \cup \{0\}$. It is also possible to construct a counterexample of the composition conjecture for moments as the following example shows. We take in equation (10) $a_1(\theta) = \sin 3\theta$ and $a_2(\theta) = \cos\theta$. In this case $\tilde{a}_1(\theta) = (1 - \cos 3\theta)/3$. It is easy to see that $\tilde{a}_1^n(\theta)$ will be a linear combination of the trigonometric functions $1, \cos 3\theta, \cos 6\theta, \ldots$ all of them orthogonal to $a_2(\theta) = \cos\theta$. Hence all the moment conditions (11) are satisfied. However the integrals

$$\int_0^{2\pi} a_1(\theta) \tilde{a}_2^n(\theta) d\theta, \tag{12}$$

with $n \in \mathbb{N} \cup \{0\}$, are in general not zero.

In [24], the characterization of all the pairs of real polynomials $p(x)$ and $q(x)$ for which the moment conditions (8) are satisfied is given. We note that this result characterizes all the Abel differential equations (5) with a center at first order of ε.

Theorem 2. [24] *Given $p(x)$ and $q(x) \in \mathbb{R}[x]$ and $a < b \in \mathbb{R}$. The moment conditions*

$$\int_a^b \tilde{q}(\sigma)\, p^n(\sigma)\, d\sigma = 0,$$

for all $n \in \mathbb{N} \cup \{0\}$ are satisfied if and only if there exist $w_1(x)$, $w_2(x)$, $\ldots$, $w_m(x) \in \mathbb{R}[x]$ with $m \geq 1$ and $w_i(a) = w_i(b)$ for $i = 1, 2, \ldots, m$ such that

$$\tilde{p}(x) = p_1(w_1(x)) = \cdots = p_m(w_m(x)) \text{ and } \tilde{q}(x) = \sum_{i=1}^{m} q_i(w_i(x)),$$

where $p_i(x)$ and $q_i(x) \in \mathbb{R}[x]$ for $i = 1, 2, \ldots, m$.

In [24] several examples are given for which the conditions stated in Theorem 2 are given and the composition condition is not satisfied.

However, in [15] it is shown that the natural translation of Theorem 2 to the trigonometric case does not hold. That is, it can be shown that there are differential equations of the form (10) with a center at first order of ε which do not satisfy the thesis of Theorem 2. The characterization of the trigonometric Abel differential equations (10) with a center at first order of ε is an open problem.

In [15] it is proved that the existence of a center in $[a, b]$ for all ε small enough of equation (5) (that is, a persistent center) implies the conditions (8) and (9). In the trigonometric case, it is also shown that if equation (10) has a persistent center then the conditions (11) and (12) need to be verified.

Recently in [27] it is proved that if conditions (8) and (9) for a polynomial Abel differential equation (5) are verified, then the composition condition is satisfied.

In the trigonometric case, that is for equation (10), if all moments conditions (11) and (12) are satisfied then equation (2) does not necessarily satisfy the composition condition, see [15]. However, under these hypothesis, it may happen that the equation (10) with $\varepsilon = 1$ has a center as the example of section 3 in [15] shows, see also [16].

The *generalized moment conditions* are

$$\int_a^b \tilde{p}^{\,n}(\sigma)\tilde{q}^{\,m}(\sigma)q(\sigma)d\sigma = 0 \quad \text{and} \quad \int_a^b \tilde{p}^{\,n}(\sigma)\tilde{q}^{\,m}(\sigma)p(\sigma)d\sigma = 0$$

for all $n, m \in \mathbb{N} \cup \{0\}$. A proof that the generalized moment conditions imply that the polynomial Abel equation satisfy the composition condition is given in [13, 26, 15]. A proof of the translation of this fact for the trigonometric Abel equation is given in [16].

In [17] the authors provide an explicit bound of the number of generalized moments (also called double moments) that have to vanish to ensure that an Abel differential equation, either in the trigonometric form (2) or in the polynomial form (3), satisfies the composition condition. This result allows to recognize the centers which satisfy the composition condition or simply *composition centers* for polynomial and trigonometric Abel differential equation. This last result is used in the next section to computationally approach the composition conjecture. In [11] the definition of *universal center* was introduced, which coincides with the definition of composition center, see [12, 20].

3. Composition conjecture

Given a polynomial Abel differential equation (3), the *center variety* is the set of polynomials $p(x)$ and $q(x)$ for which the equation has a center in $[a, b]$ and the *composition center variety* is the set of polynomials $p(x)$ and $q(x)$ for which the equation has a composition center (that is the composition condition is verified) in $[a, b]$. After all that we have said in the previous sections the statement of the composition conjecture is the following.

Conjecture 3. *For any polynomial Abel differential equation* (3) *the center variety and the composition center variety coincide.*

We recall that this conjecture is not true for trigonometric Abel differential equations, see [1, 3, 15, 20, 21, 22]. Moreover in [22] was proved that the lowest degree of a trigonometric Abel differential equation (2) with a non-composition center is 3. Conjecture 3 is satisfied under certain restrictions of the coefficients of the polynomial Abel differential equation, see for instance Theorem 2 in [3] and Theorem 2 in [6].

However a systematic verification of Conjecture 3 has not been done. The aim of this section is to verify if all the centers of the polynomial Abel differential equation (3) for lower degrees of p and q are composition centers. We also analyze the case in which the number of monomials in $p(x)$ and $q(x)$ is up to 2.

As we have said, in [16] another characterization of the composition centers is provided in terms of the vanishing of a finite set of generalized moments or double moments. As usual for a polynomial $p(x) \in \mathbb{R}[x]$, δp denotes the degree of p.

Theorem 4. [17] *Given* $p, q \in \mathbb{R}[x]$ *with* $\max(\delta p, \delta q) = n$, *equation* (3) *has a composition center if and only if for all* $i, j \in \mathbb{N} \cup \{0\}$ *satisfying* $i + j \leq 2n - 3$,

$$\int_a^b \tilde{p}^i(x)\tilde{q}^j(x)q(x)\,dx \;=\; \int_a^b p(x)\,dx \;=\; 0. \tag{13}$$

This characterization of the composition centers allows to discriminate the composition centers from other centers and approaches the conjecture from a computational point of view. The main results of the paper are the following.

Theorem 5. *For any polynomial Abel differential equation with degree* $\max(\delta p, \delta q) \leq 3$ *the center variety and the composition center variety coincide.*

The proof of Theorem 5 is given in section 4.

We have also dealt with the case in which $\max(\delta p, \delta q) = 4$. In this case we cannot end up with all the computations to ensure that the center variety and the composition center variety coincide. In section 5, we will make use of modular arithmetics and the algorithm described in [28] which provide the center variety with a probability close to 1.

All the pairs $p(x)$ and $q(x)$ that we find using this algorithm give rise to composition centers. Therefore, we can state the following conjecture.

Conjecture 6. *For any polynomial Abel differential equation with degree* $\max(\delta p, \delta q) = 4$ *the center variety and the composition center variety coincide.*

The computations motivating this conjecture are given in section 5.

Given a polynomial Abel differential equation (3) defined in the real interval $[a, b]$, with $a < b$, we can make the following change of the independent variable $x \to (x - a)/(b - a)$. This leads to an Abel differential equation defined on the real interval $[0, 1]$.

Theorem 7. *Consider a polynomial Abel differential equation* (3) *defined on the real interval* $[0, 1]$. *Assume that* p *and* q *only have two monomials, that is,*

$$p(x) = a_i x^i + a_j x^j \quad \text{and} \quad q(x) = a_m x^m + a_n x^n,$$

with $a_i, a_j, a_m, a_n \in \mathbb{R}$ *and* $i, j, m, n \in \mathbb{N} \cup \{0\}$. *Then the center variety and the composition center variety coincide.*

The proof of Theorem 7 is given in section 6.

4. Proof of Theorem 5

Given an ordinary differential equation of the form (3), there is a well known general method to compute center conditions which was proved by Poincaré. We will denote the center conditions as the *Poincaré–Liapunov constants* for equation (3). In order to compute them we propose a formal first integral of the form $H(y, x) = y + \sum_{k=2}^{\infty} h_k(x) y^k$, where $h_k(x)$ are polynomials. We recall that a first integral for an equation (3) satisfies that $\dot{H} = \dot{y}\, \partial H/\partial y + \dot{x}\, \partial H/\partial x \equiv 0$, where $\dot{y} = p(x)y^2 + q(x)y^3$, $\dot{x} = 1$. By imposing that $\dot{H} = 0$, we obtain the following recursive system of linear differential equations

$$h_k'(x) + (k - 1)p(x)h_{k-1}(x) + (k - 2)q(x)h_{k-2}(x) = 0, \tag{14}$$

for $k \geq 2$ and with $h_0(x) \equiv 0$ and $h_1(x) \equiv 1$. From the recursive system (14) we compute the polynomials $h_k(x)$ and we obtain the Poincaré–Liapunov constant $v_k := h_k(b) - h_k(a)$. The equation has a center in $[a, b]$ if $v_k = 0$ for all $k \geq 2$. We note that v_k is a polynomial in the coefficients of $p(x)$ and $q(x)$.

We denote the coefficients of $p(x)$ and $q(x)$ in the following way $p(x) = \sum_{i=0}^{3} b_i x^i$, $q(x) = \sum_{i=0}^{3} c_i x^i$.

In order to proof the result we have computed fifteen necessary conditions $v_k = 0$ for $k = 2, \ldots, 16$. These necessary conditions are very long, so we do not present them here. However, one can check our computations with the help of any available computer algebra system. In this case, in order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle v_2, v_3, \ldots, v_{16} \rangle$. This is an extremely difficult computational problem. We have used the routine *minAssGTZ* of the computer algebra system Singular [23] and we have found the irreducible decomposition of the variety of the ideal I over the field of rational numbers when $\max(\delta p, \delta q) \leq 3$.

The obtained decomposition consists of 2 components defined by the following ideals

1) $\langle 2c_2 + 3c_3, 4c_0 + 2c_1 - c_3, 2b_2 + 3b_3, 4b_0 + 2b_1 - b_3 \rangle$;
2) $\langle b_3c_2 - b_2c_3, b_3c_1 - b_1c_3, b_2c_1 - b_1c_2, 12c_0 + 6c_1 + 4c_2 + 3c_3, 12b_0 + 6b_1 + 4b_2 + 3b_3 \rangle$;

The generalized moment conditions u_i are obtained computing the integrals (13) and in this case we have found the irreducible decomposition of the variety of the ideal $J = \langle u_1, u_2, \ldots, u_{17} \rangle$ over the field of rational numbers. To deduce if all the centers are composition centers we must only compare both decompositions and in both cases they are the same.

5. On the Conjecture 6

As before, we denote the coefficients of $p(x)$ and $q(x)$ in the following way $p(x) = \sum_{i=0}^{4} b_i x^i$, $q(x) = \sum_{i=0}^{4} c_i x^i$. We have computed fifteen necessary conditions $v_k = 0$ for $k = 2, \ldots, 16$, that we do not present here. In order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ as in the previous section. In this case, however, we cannot find the irreducible decomposition of the variety of the ideal I over the field of rational numbers due to the computational difficulty. We try to find this irreducible decomposition over a finite field. We take the prime $p = 32003$ and we have found this decomposition over the finite field $\mathbb{Z}/(p)$. We have chosen this prime because the algorithm turned out to be very efficient and goes to a reasonable speed when using it.

We have followed the algorithm described in [28] which makes use of modular arithmetics. The modular approach used to obtain center conditions consists on the following five steps.

Step 1. Choose a prime number p and from the ideal I compute the minimal associated primes $\tilde{I}_1, \ldots, \tilde{I}_s$ with coefficients in $\mathbb{Z}_p$,

Step 2. Using the rational reconstruction algorithm of Wang et al. [29], we obtain the ideals I_i, $i = 1, \ldots, s$, with coefficients in $\mathbb{Q}$, Step 3. For each i, using the radical membership test, check whether the polynomials v_k for $k = 2, \ldots, 16$ are in the radicals of the ideals I_i, that is, whether the reduced Gröbner basis of the ideal $< 1 - wv_j, I_i >$ is equal to $\{1\}$, where w is a mute variable. If yes, then go to *Step 4*, otherwise take another prime p and go to *Step 1*.

Step 4. Compute the intersection over the rational numbers $Q = \cap_{i=1}^{s} I_i$,

Step 5. Check that $\sqrt{Q} = \sqrt{I}$, that is, that for any $q_i \in Q$, the reduced Gröbner basis of the ideal $< 1 - wq_i, I >$ is equal to $\{1\}$ and for any $v_j \in I$, the reduced Gröbner basis of the ideal $< 1 - wv_j, Q >$ is equal to $\{1\}$. Recall that $I = \langle v_2, v_3, \ldots, v_{16} \rangle$. If this is the case, then $V(I) = \cup_{i=1}^{s} V(I_i)$. If not, then go to *Step 1* and choose another prime p.

We note that whenever we compute the Gröbner basis of an ideal, we must to do it over the field of rational numbers.

The last step of this algorithm has not been verified into the field of rational numbers. However, we have checked it over finite fields $\mathbb{Z}/(p)$, with different prime numbers p. This last step ensures that all the points of the variety $V(I)$ have been found. That is, we know that all the encountered points belong to the decomposition of $V(I)$ but we do not know whether the given decomposition is complete. We remark that, nevertheless, it is practically sure that the given list is complete, see for instance [5, 19, 28]. Therefore, in the following we provide sufficient conditions to have a center, which are practically necessary. We denote this situation by the expression *with probability close to* 1.

The obtained decomposition for the case $\max(\delta p, \delta q) = 4$ consists of 2 components defined by the following ideals

1) $\langle c_4, 2c_2 + 3c_3, 4c_0 + 2c_1 - c_3, b_4, 2b_2 + 3b_3, 4b_0 + 2b_1 - b_3 \rangle$;
2) $\langle b_4c_3 - b_3c_4, b_4c_2 - b_2c_4, b_3c_2 - b_2c_3, b_4c_1 - b_1c_4, b_3c_1 - b_1c_3, b_2c_1 - b_1c_2, 60c_0 + 30c_1 + 20c_2 + 15c_3 + 12c_4, 60b_0 + 30b_1 + 20b_2 + 15b_3 + 12b_4 \rangle$;

The generalized moment conditions u_i are obtained computing the integrals (13) and in this case we have found the irreducible decomposition of the variety of the ideal $J = \langle u_1, u_2, \ldots, u_{17} \rangle$ over the field of rational numbers. To deduce if all the centers are composition centers we must only compare both decompositions and in both cases they are the same.

For the case $\max(\delta p, \delta q) = 5$ without loss of generality we can divide the study in two cases: either $b_5 = 1$ or $b_5 = 0$ and $c_5 = \pm 1$. We get these cases by a rescaling of the form $y = kY$ with $k \neq 0$. Even in the simple case that $b_5 = 0$ and $c_5 = \pm 1$, and using modular arithmetics, we have not been able to find the irreducible decomposition of the variety $V(\langle v_2, v_3, \ldots, v_{16}\rangle)$.

6. Proof of Theorem 7

In equation (3) we write $p(x) = a_i x^i + a_j x^j$ and $q(x) = a_m x^m + a_n x^n$. Using the method of construction of a formal first integral described at the beginning of section 4, we obtain that the first Poincaré–Liapunov constant is $v_2 = -(1+j)a_i - (1+i)a_j$. All the Poincaré–Liapunov constants computed in this section have been obtained by using the algorithm described in the appendix. The vanish of v_2 gives us $a_i = (1+i)a_j/(1+j)$. The second Poincaré–Liapunov constant is $v_3 = -(1+n)a_m - (1+m)a_n$. Vanishing this constant we obtain $a_n = (1+n)a_m/(1+m)$. We note that at this moment we have that $\tilde{p}(x) = \int_0^x p(\sigma)d\sigma$ and $\tilde{q}(x) = \int_0^x q(\sigma)d\sigma$ satisfy that $\tilde{p}(0) = \tilde{p}(1) = 0$ and $\tilde{q}(0) = \tilde{q}(1) = 0$.

The third Poincaré–Liapunov constant is given by $v_4 = -a_j a_m (i-j)(m-n)(i+j+ij-m-n-mn)$. We divide the study of the vanishing of v_4 in three cases.

First case: $a_j a_m = 0$. When $a_j = 0$ we have that $a_i = 0$ and then $p(x) \equiv 0$. This case gives a differential equation with separated variables which forms a composition center (recall that $\tilde{q}(0) = \tilde{q}(1) = 0$). In the case that $a_m = 0$ we get an analogous result.

Second case: $(i-j)(m-n) = 0$. If $i = j$ then $p(x)$ has a single monomial $p(x) = a_j x^j$ and then $\tilde{p}(x) = a_j x^{j+1}/(j+1)$. The condition $\tilde{p}(1) = 0$ implies that $a_j = 0$ and, hence, $p(x) \equiv 0$. We get again a differential equation with separated variables which forms a composition center (recall that $\tilde{q}(0) = \tilde{q}(1) = 0$). In the case that $m = n$ we get an analogous result.

Third case: $i + j + ij - m - n - mn = 0$. We take $i = (-j + m + n + mn)/(1+j)$. The next Poincaré–Liapunov constant is

$$v_5 = -2a_j^2 a_m (j-m)(j-n)(m-n)(2j + j^2 - m - n - mn)^2.$$

Excluding the previous cases, we get that either $j = n$ or $(2j + j^2 - m - n - mn) = 0$. In the case that $j = n$ we get that there exists a constant C such that $p(x) = Cq(x)$ which forms a composition center. In the latter case $(2j + j^2 - m - n - mn) = 0$, we obtain that $i = j$ and, thus, $p(x) \equiv 0$.

7. Appendix

Program to compute the Poincaré–Liapunov constants for an equation (3) defined in the interval $[0,1]$ and with p and q polynomials up to degree 5.

$p = b0 + b1x + b2x^2 + b3x^3 + b4x^4 + b5x^5;$
$q = c0 + c1x + c2x^2 + c3x^3 + c4x^4 + c5x^5;$
$h = -Apply[Plus, Integrate[Cases[Expand[p], \_], x]];$
$Numerator[Factor[(h/.x- > 1) - (h/.x- > 0)]] >> v2.txt$
$hh = h; h = -Apply[Plus, Integrate[Cases[Expand[2 * p * hh + q], \_], x]];$
$Numerator[Factor[(h/.x- > 1) - (h/.x- > 0)]] >> v3.txt$
$For[k = 4, k < 16, k + +, hhh = hh; hh = h;$
$h = -Apply[Plus, Integrate[Cases[Expand[(k - 1) * p * hh$
$+(k - 2) * q * hhh], \_], x]];$
$Put[Numerator[Factor[(h/.x- > 1) - (h/.x- > 0)]],$
$StringJoin["v", ToString[k], ".txt"]]]$

Program to compute the moment conditions for an equation (3) defined on $[0,1]$.

$p = b0 + b1x + b2x^2 + b3x^3 + b4x^4 + b5x^5;$
$q = c0 + c1x + c2x^2 + c3x^3 + c4x^4 + c5x^5;$
$P = Integrate[p/.x- > z, z, 0, x];$
$Q = Integrate[q/.x- > z, z, 0, x];$
$Numerator[Factor[Integrate[p, x, 0, 1]]] >> u1.txt;$
$Numerator[Factor[Integrate[q, x, 0, 1]]] >> u2.txt;$
$For[i = 1, i < 8, i + +, For[j = 1, j < i + 1, j + +,$
$Put[Numerator[Factor[Apply[Plus, Integrate[Cases[$
$Expand[P^\wedge(i - j) * Q^\wedge j * p], \_], x, 0, 1]]]],$
$StringJoin["u", ToString[2 + (i - 1)i/2 + j], ".txt"]]]]$

References

[1] M.A.M. Alwash, *On a condition for a center of cubic non-autonomous equations*, Proc. Roy. Soc. Edinburgh Sect. A **113** (1989), 289–291.

[2] M.A.M. Alwash, *On the composition conjectures*, Electron. J. Differential Equations **2003**, No. 69, 4 pp.

[3] M.A.M. Alwash, *The composition conjecture for Abel equation*, Expo. Math. **27** (2009), no. 3, 241–250.

[4] M.A.M. Alwash, N.G. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh Sect. A **105** (1986), 129–152.

[5] E.A. Arnold, *Modular algorithms for computing Gröbner bases*, J. Symbolic Comput. 35 (4) (2003) 403–419.

[6] M. Blinov, Y. Yomdin, *Generalized center conditions and multiplicities for polinomial Abel equations of small degrees*, Nonlinearity **12** (1999), 1013–1028.

[7] M. Briskin, J.P. Françoise, Y. Yomdin, *Center conditions, compositions of polynomials and moments on algebraic curves*, Ergodic Theory Dynam. Systems **19** (1999), 1201–1220.

[8] M. Briskin, J.P. Françoise, Y. Yomdin, *Center conditions. II. Parametric and model center problems*, Israel J. Math. **118** (2000), 61–82.

[9] M. Briskin, J.P. Françoise, Y. Yomdin, *Center conditions. III. Parametric and model center problems*, Israel J. Math. **118** (2000), 83–108.

[10] M. Briskin, N. Roytvarf, Y. Yomdin, *Center conditions at infinity for Abel differential equations*, Ann. of Math. (2) **172** (2010), no. 1, 437–483.

[11] A. Brudnyi, *An algebraic model for the center problem*, Bull. Sci. Math. **128** (2004), no. 10, 839–857.

[12] A. Brudnyi, *On the center problem for ordinary differential equations*, Amer. J. Math. **128** (2006), 419–451.

[13] A. Brudnyi, Y. Yomdin, *Tree composition condition and moments vanishing*, Nonlinearity **23** (2010) 1651–1673.

[14] L.A. Cherkas, *The number of limit cycles of a certain second order autonumous system* (Russian), Differencial'nye Uravnenija **12** (1976), no. 5, 944-946. Differential Equations **12** (1976), no. 5, 666–668.

[15] A. Cima, A. Gasull, F. Mañosas, *Centers for trigonometric Abel equations*, Qual. Theory Dyn. Syst. **11** (2012), no. 1, 1937.

[16] A. Cima, A. Gasull, F. Mañosas, *A simple solution of some conjectures for Abel equations*, J. Math. Anal. Appl. **398** (2013), 477486.

[17] A. Cima, A. Gasull, F. Mañosas, *An explicit bound of the number of vanishing double moments forcing composition*, J. Differential Equations **255** (2013), no. 3, 339–350.

[18] J. Devlin, N.G. Lloyd, J.M. Pearson,*Cubic systems and Abel equations.* J. Differential Equations **147** (1998), no. 2, 435–454.

[19] H-C. Graf von Bothmer, *Experimental results for the Poincaré center problem*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), no. 5-6, 671–698.

[20] J. Giné, M. Grau, J. Llibre, *Universal centers and composition conditions*, Proc. Lond. Math. Soc. (3) **106** (2013), 481–507.

[21] J. Giné, M. Grau, X. Santallusia, *Composition conditions in the trigonometric Abel equations*, J. Appl. Anal. Comput. **3** (2013), no. 2, 133–144.

[22] J. Giné, M. Grau, X. Santallusia, *Universal centers in the cubic trigonometric Abel equation*, Electron. J. Qual. Theory Differ. Equ. **2014**, No. 1, 1–7.

[23] G.M. Greuel, G. Pfister, H. Schönemann, Singular 3.0 A computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, 2005. `http://www.singular.uni-kl.de`

[24] M. Muzychuk, F. Pakovich, *Solution of the polynomial moment problem*, Proc. Lond. Math. Soc. (3) **99** (2009), 633–657.

[25] F. Pakovich, *A counterexample to the composition conjecture*, Proc. Amer. Math. Soc, **130** (2002), 3747–3749.

[26] F. Pakovich, *On the polynomial moment problem*, Math. Res. Lett. **10** (2003) 401–410.

[27] F. Pakovich, *Solution of the parametric center problem for the Abel differential equation*, preprint, arXiv:1407.0150 [math.CA].

[28] V.G. Romanovski, M. Prešern, *An approach to solving systems of polynomials via modular arithmetics with applications*, J. Comput. Appl. Math. 236 (2011), no. 2, 196–208.

[29] P.S. Wang, M.J.T. Guy, J.H. Davenport, *P-adic reconstruction of rational numbers* SIGSAM Bull. **16**(1982), no. 2, 2–3.

Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69. 25001 Lleida, Catalonia, Spain

E-mail address: `gine@matematica.udl.cat`

E-mail address: `mtgrau@matematica.udl.cat`

E-mail address: `xivaer@gmail.com`